Spectral Projection Methods for Integral and Integro-Differential Equations

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Abstract

Integral and integro-differential equations are widely used in various scientific and engineering fields. Finding highly accurate approximate solutions often poses significant challenges, especially when dealing with singularities in the kernel and non-smooth solutions. This thesis explores the development of spectral projection methods to find approximate solutions for various types of linear and nonlinear integral and integro-differential equations, as well as eigenvalue problems of the integral operators.

We develop Jacobi spectral Galerkin approach for finding approximate solutions of the linear Fredholm integral equations with an algebraic weakly singular kernel. The exact solution to this type of equation is generally non-smooth. Our approach incorporates the singular part of the kernel into a single Jacobi weight function. This enables us to achieve superconvergence rates in both supremum and weighted L^2 norms for non-smooth solutions using the iterated Jacobi spectral Galerkin method. We also propose Jacobi spectral multi-Galerkin method and its iterated version, based on the Galerkin approach and obtain superconvergence results. Additionally, we improve the convergence rate of all the proposed methods by improving the regularity of exact solution, using a smoothing transformation. Further, we apply our proposed Jacobi spectral Galerkin approach to address the eigenvalue problem of Fredholm integral operators with an algebraic weakly singular kernel on a Banach space. Using this approach, we achieve superconvergence rates for the approximate eigenvalues obtained by the Jacobi spectral multi-Galerkin method. Superconvergence for the approximate eigenfunctions is obtained by the iterated Jacobi spectral multi-Galerkin method.

Next, we investigate a class of nonlinear Fredholm integro-differential equations with smooth kernel. We find that the Legendre spectral multi-Galerkin, multi-collocation, and iterated version of Galerkin, multi-Galerkin and multi-collocation methods yield superconvergence results, despite the presence of unbounded differential operator. Further, we explore the nonlinear weakly singular Fredholm integral equations of Hammerstein type. Our proposed iterated Jacobi spectral Galerkin method achieves similar superconvergence results for non-smooth solutions, as observed in the linear case. Additionally, we demonstrate the applicability of the Jacobi spectral Galerkin approach for a class of high-order nonlinear weakly singular integro-differential equations, achieving superconvergence.

Finally, we consider multi-term algebraic weakly singular Volterra integral equations and propose Chebyshev spectral Galerkin, collocation, multi-Galerkin, and multi-collocation methods to obtain approximate solutions. We achieve superconvergence for the Chebyshev

spectral multi-Galerkin and multi-collocation methods after the smoothing transformation and obtain sharp error bounds in the supremum norm by leveraging the behaviour of the exact solution.

Numerical implementations of all the proposed methods are carried out to verify the theoretical results.

List of Publications/Pre-prints from the Thesis

1. Mandal, M., **Kayal, A.**, & Nelakanti, G. (2023). Projection methods for approximate solution of a class of nonlinear Fredholm integro-differential equations. **Applied Numerical Mathematics**, 184, 49-76.

2. **Kayal, A.**, & Mandal, M. (2024). A new approach of shifted Jacobi spectral Galerkin methods (SJSGM) for weakly singular Fredholm integral equation with non-smooth solution. **Numerical Algorithms**, 96(4), 1553-1582.

3. **Kayal, A.**, & Mandal, M. (2025). Superconvergent method for weakly singular Fredholm-Hammerstein integral equations with non-smooth solutions and its application. **Applied Numerical Mathematics**, 207, 24-44.

4. **Kayal, A.**, & Mandal, M. Jacobi spectral Galerkin methods for eigenvalue problems of weakly singular integral operators. (**Under Review**)

5. **Kayal, A.**, & Mandal, M. Chebyshev spectral projection methods for multi-term weakly singular Volterra integral equations. (**Under Review**)

Other Publications

1. **Kayal, A.**, Mandal, M., & Nelakanti, G. (2024). Superconvergence of Legendre spectral projection methods for mth order integro-differential equations with weakly singular kernels. **Journal of Computational and Applied Mathematics**, 439, 115585.

2. Gupta, S., **Kayal, A.**, & Mandal, M. (2025). Superconvergence results for hyper-singular integral equation of first kind by Chebyshev spectral projection methods. **Applied Mathematics and Computation**, 487, 129093.